Chapter 2

Nature of Solar Radiation

Solar energy comes to Earth in the form of radiation, or sunlight, with spectral components mostly in the visible, near infrared, and near ultraviolet. To study the properties of sunlight, we need to consider and understand it from two points of view: as an electromagnetic wave and as a flow of photons. The first point of view is essential for all solar thermal applications and the antireflection coatings for solar cells. The second point of view is essential with regard to solar cells and solar photochemistry. The unification of the two points of view is represented by quantum electrodynamics, one of the most fruitful and matured fields in modern physics. Here, for simplicity, we will present an elementary treatment of these two points of view separately.

2.1 Light as Electromagnetic Waves

Up to the middle of the nineteenth century, electromagnetic phenomena and light have been considered as totally independent entities. In 1865, in a monumental paper *A Dynamic Theory of the Electromagnetic Field*, James Clerk Maxwell (Fig 2.1) proposed that light is an electromagnetic wave [58]. In that paper, he developed a complete set of

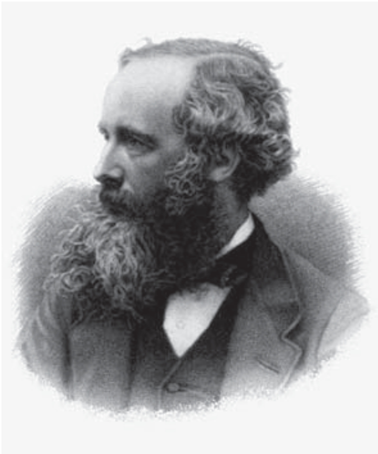


Figure 2.1 James Clerk Maxwell. Scottish physicist (1831–1879), one of the most influential physicists along with Isaac Newton and Albert Einstein. He developed a set of equations describing electromagnetism, known as the *Maxwell's equations*. In 1865, based on those equations, he predicted the existence of electromagnetic waves and proposed that light is an electromagnetic wave [58]. He also pioneered the kinetic theory of gases, and created a science fiction character *Maxwell's demon*. Portrait courtesy of Smithsonian Museum.

equations explaining electromagnetic phenomena, now known as *Maxwell's equations*. Based on those equations, he predicted the existence of electromagnetic waves, propagating in free space with a speed that equals exactly the speed of light, which was then verified experimentally by Heinrich Hertz. Maxwell's bold postulation that light is an electromagnetic wave has since become one of the cornerstones of physics.

2.1.1 Maxwell's Equations

In vacuum, or free space, Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_{\mathbf{0}}}, \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = \mathbf{0}, \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.3}$$

$$\nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}. \tag{2.4}$$

Electric current cannot exist in free space. For linear, uniform, isotropic materials, the current density $\mathbf{J}$ is determined by the electric field intensity $\mathbf{E}$ through Ohm's law,

$$\mathbf{J} = \sigma \mathbf{E}. \tag{2.5}$$

The names, meanings, and units of the physical quantities in these equations are listed in Table 2.1. For example, the electric constant has an intuitive meaning as follows. A capacitor made of two parallel conducting plates with area A and distance d has a capacitance $C = \varepsilon_0 A/d$ in farads. Similarly, the electric constant has an intuitive meaning as follows. An inductor made of a long solenoid of N loops with cross-sectional area A and length l has an inductance $L = \mu_0 N^2 A/l$ in henrys.

Table 2.1: Quantities in Maxwell's Equations

Symbol	Name	Unit	Meaning or Value
$\mathbf{E}$	Electric field intensity	V/m	
$\mathbf{B}$	Magnetic field intensity	T (tesla)	N/A·m
ρ	Electric charge density	C/m^3	
$\mathbf{J}$	Electric current density	A/m^2	
ε_0	Electric constant (permittivity of free space)	F/m	8.85×10^{-12} F/m
μ_0	Magnetic constant (permeability of free space)	H/m	$4\pi \times 10^{-7}$ H/m
σ	Conductivity	$(\Omega \cdot m)^{-1}$	

2.1.2 Vector Potential

To treat the electromagnetic field in space, a convenient method is to use the *vector potential*. From Eq. 2.2, it is possible to construct a vector field $\mathbf{A}$ which satisfies

$$\mathbf{B} = \nabla\times\mathbf{A}. \tag{2.6}$$

Then, Eq. 2.2 is automatically satisfied. Substituting Eq. 2.6 into Eq. 2.3, one obtains

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\nabla\times\mathbf{B}. \tag{2.7}$$

For any function $\phi(\mathbf{r})$, $\nabla \times [\nabla\phi(\mathbf{r})] = \mathbf{0}$, it is possible to set up the vector potential $\mathbf{A}$ such that

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \tag{2.8}$$

where ϕ is the electrostatic potential arising from the charges. The choice of the vector potential is not unique. By adding a gradient of an arbitrary function to it, values of the electric field and magnetic field do not change. This is called the *gauge invariance* of the vector potential. It is possible to define a vector potential which satisfies the condition

$$\nabla \cdot \mathbf{A} = 0. \tag{2.9}$$

Equation 2.9 is called the *Coulomb gauge*, which is the most convenient gauge to treat nonrelativistic problems of an atomic system and an independent electromagnetic wave. In fact, using Eq. 2.9 and the first Maxwell equation Eq. 2.1, one obtains

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}, \tag{2.10}$$

which means that the scalar potential is generated by the static charges only. It is thus convenient for treating the problems of interactions between the radiation field and atomic systems. For details of the gauge problem, see, for example, *The Quantum Theory of Radiation* by Walter Heitler [37].

2.1.3 Electromagnetic Waves

In this section, we study the electromagnetic waves in free space, that is, where the electric charge ρ and current $\mathbf{J}$ are zero. Substituting Eqs 2.6 and 2.8 into Eq. 2.4, we have

$$\nabla \times \nabla \times \mathbf{A} + \varepsilon_0\mu_0\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{2.11}$$

Using the identity

$$\nabla \times \nabla \times \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^{\mathbf{2}}\mathbf{A} \tag{2.12}$$

and Eq. 2.9, Eq. 2.11 becomes

$$\nabla^2\mathbf{A} - \varepsilon_0\mu_0\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \tag{2.13}$$

Introducing

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}, \tag{2.14}$$

Eq. 2.13 becomes

$$\nabla^2 \mathbf{A} - \frac{\mathbf{1}}{\mathrm{c}^2} \frac{\partial^2 \mathbf{A}}{\partial \mathbf{t^2}} = \mathbf{0}, \tag{2.15}$$

which is a wave equation with velocity c. Because of Eqs. 2.6 and 2.8, the electric field intensity and the magnetic field intensity also satisfy the same wave equation,

$$\nabla^2 \mathbf{E} - \frac{1}{\mathrm{c}^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \tag{2.16}$$

and

$$\nabla^2 \mathbf{B} - \frac{1}{\mathrm{c}^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0, \tag{2.17}$$

According to values of ε_0 and μ_0 coming from electromagnetic measurements in 1860s, the velocity of electromagnetic waves should be 3.1×10^8 m/s. On the other hand, experimental values of the speed of light at that time were 2.98×10^8–3.15×10^8 m/s. The difference was within experimental error. Maxwell proposed thusly [58]:

> The agreement of the results seems to show that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field according to electromagnetic laws.

Maxwell's theory of electromagnetic waves was experimentally verified by Heinrich Hertz in 1865. From recent electrical measurements, one finds $1/\sqrt{\varepsilon_0 \mu_0} = 2.998 \times 10^8$m/s, which is exactly the speed of light in a vacuum, c.

2.1.4 Plane Waves

An electromagnetic wave with circular frequency ω in space is defined as

$$\mathbf{A}(x, y, z, t) = \mathbf{A}(x, y, z)\, e^{-i\omega t}. \tag{2.18}$$

To study the properties of electromagnetic waves, we consider the case that the wave propagates in one direction, say z. In this case, the field intensities only depend on z. Equation 2.15 becomes

$$\frac{d^2 \mathbf{A}}{dz^2} + \frac{\omega^2}{c^2} \mathbf{A} = \mathbf{0}. \tag{2.19}$$

The general solution is

$$\mathbf{A} = \mathbf{A_0} e^{i(k_z z - \omega t)}. \tag{2.20}$$

where $\mathbf{A}_0$ is a constant, and the z-component of the *wavevector* k_z is defined as

$$k_z = \frac{\omega}{c}. \tag{2.21}$$

2.1.5 Polarization of Light

Although in general the vector potential could have x, y, z-components, because of Eq. 2.9, the z-component of the vector potential must be zero,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = ik_z A_z = 0. \tag{2.22}$$

This means that A_z must be a constant over the entire space. Because we are interested in electromagnetic waves, or the variation of electromagnetic fields, we can simply set $A_z = 0$. The waves are *transverse*. In other words, the intensity vectors are perpendicular to the direction of propagation.

The direction of the vector potential could be either x or y or a linear combination of x- and y-components. For the x-component of $\mathbf{A}$, we have

$$A_x = A_{x0} e^{i(k_z z - \omega t)}, \quad A_y = 0, \quad A_z = 0. \tag{2.23}$$

The electric field intensity, according to Eq. 2.8, is

$$E_x = i\omega A_{x0}\, e^{i(k_z z - \omega t)}, \quad E_y = 0, \quad E_z = 0. \tag{2.24}$$

And the magnetic field intensity, according to Eq. 2.6, is

$$B_x = 0, \quad B_y = ik_z A_{x0}\, e^{i(k_z z - \omega t)}, \quad B_z = 0. \tag{2.25}$$

Therefore, the only nonvanishing components of the electric field intensity and the magnetic field intensity are E_x and B_y. According to Eq. 2.21, they are in phase and proportional,

$$E_x = c\, B_y. \tag{2.26}$$

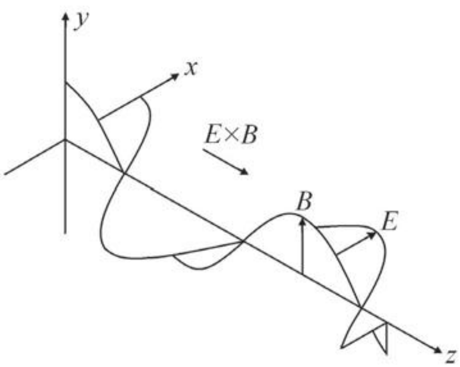


Figure 2.2 Electromagnetic wave. The electromagnetic wave is *transverse*, where the intensity vectors $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to the direction of propagation. The electric field intensity $\mathbf{E}$ is perpendicular to the magnetic field intensity $\mathbf{B}$. The energy flux vector $\mathbf{S} = \mu_0^{-1}\mathbf{E} \times \mathbf{B}$ is formed from $\mathbf{E}$ and $\mathbf{B}$ by a right-hand rule.

In summary, according to the electromagnetic theory of light, the electric field intensity vector is perpendicular to the direction of the propagation of light. The magnetic field intensity is perpendicular to both the direction of the electric field intensity vector and the direction of the propagation of light, and its magnitude is proportional to the electric field intensity. See Fig. 2.2.

2.1.6 Motion of an Electron in Electric and Magnetic Fields

In this section, the interaction of the radiation field—electric and magnetic fields varying with time—with the electrons is studied using classical mechanics as a preparation for a quantum-mechanical treatment.

The standard method of developing the quantum mechanics of a dynamic system is first to cast the classical equation of motion into Hamiltonian format. The Hamiltonian $H(\mathbf{p}, \mathbf{r})$ of a dynamic system is a function of its coordinate $\mathbf{r}$ and corresponding momentum $\mathbf{p}$, representing the total energy. For example, for an electron with charge q moving in an electric field with potential $\phi(\mathbf{r})$, the Hamiltonian is

$$H = \frac{1}{2m_e}\mathbf{p}^2 + q\phi(\mathbf{r}). \tag{2.27}$$

The equations of motion in the Hamiltonian format are a pair of first-order ordinary differential equations:

$$\dot{p}_x = -\frac{\partial H}{\partial x}, \tag{2.28}$$

$$\dot{x} = \frac{\partial H}{\partial p_x}. \tag{2.29}$$

There are similar equations for y and z. Using Eqs 2.29 and 2.27, the expression of momentum is found to be identical to the usual definition,

$$\mathbf{p} = m_e\mathbf{v} = m_e\dot{\mathbf{r}}, \tag{2.30}$$

where a dot means taking a derivative with respect to time t. Applying Eqs. 2.28 and 2.29 to Eq. 2.27, one finds

$$m_e\ddot{\mathbf{r}} = -q\nabla\phi(\mathbf{r}) = q\mathbf{E}, \tag{2.31}$$

which is Newton's equation of motion, where $\mathbf{E}$ is electric field intensity.

For the motion of an electron in both electric and magnetic fields, the standard method is to insert the vector potential $\mathbf{A}$ into the expression of momentum by simply substituting $\mathbf{p}$ with $\mathbf{p} - q\mathbf{A}$ in the Hamiltonian,

$$H = \frac{1}{2m_e}(\mathbf{p} - q\mathbf{A})^2 + q\phi(\mathbf{r}), \tag{2.32}$$

or

$$H = \frac{1}{2m_e}\left[(p_x - qA_x)^2 + (p_y - qA_y)^2 + (p_z - qA_z)^2\right] + q\phi(x, y, z). \tag{2.33}$$

Applying Eq. 2.29 to Eq. 2.32, one obtains

$$p_x = m_e\dot{x} + qA_x \tag{2.34}$$

and so on. In vector form, it is

$$\mathbf{p} = m_e\dot{\mathbf{r}} + q\mathbf{A}, \tag{2.35}$$

which is the definition of momentum in a magnetic field. Applying Eq. 2.28 to Eq. 2.32 and using Eq. 2.35, for the x-component, yield

$$\frac{dp_x}{dt} = q\left[\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}\right] - q\frac{\partial \phi}{\partial x}. \tag{2.36}$$

The familiar Newton equation of motion, similar to Eq. 2.31, can be obtained from Eqs. 2.33 and 2.36. The x-component is given as

$$m_e\ddot{x} = \frac{dp_x}{dt} - q\frac{dA_x}{dt}. \tag{2.37}$$

Note that

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}, \tag{2.38}$$

for example, for the x-component, one obtains,

$$m_e\ddot{x} = q\left[\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \dot{z}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) - \frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x}\right]. \tag{2.39}$$

Using Eqs. 2.6 and 2.8, in vector form, the equation of motion is

$$m_e\ddot{\mathbf{r}} = q\mathbf{E} + q\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}) = q\mathbf{E} + q\dot{\mathbf{r}} \times \mathbf{B}. \tag{2.40}$$

which is Newton's equation of motion including the magnetic force. Therefore, the correctness of the Hamiltonian, Eq. 2.32, is verified. We will use the Hamiltonian in the quantum-mechanical treatment of the interaction of radiation with atomic systems.

2.2 Optics of Thin Films

Maxwell's theory of light plays a critical role in the understanding of selective absorption films for solar thermal applications and antireflection films in photovoltaics. The general theory with an arbitrary incident angle is rather complicated. However, for applications related to solar energy, it suffices to study the case of normal incidence, which demonstrates most of the related physics. First, let us extend Maxwell's equations to dielectrics.

2.2.1 Relative Dielectric Constant and Refractive Index

Maxwell's equations, Eqs. 2.1–2.4 are used for the case of a vacuum. To describe electromagnetic phenomena in a nonmagnetic medium, the electric constant ε_0 is replaced by the electric constant of the medium, ε. Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon}, \tag{2.41}$$

$$\nabla \cdot \mathbf{B} = \mathbf{0}, \tag{2.42}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.43}$$

$$\nabla \times \mathbf{B} = \varepsilon \mu_{\mathbf{0}} \frac{\partial \mathbf{E}}{\partial t} + \mu_{\mathbf{0}} \mathbf{J}. \tag{2.44}$$

Following the procedures in Section 2.1.1, we found the wave equations for the electric field intensity and the magnetic field intensity:

$$\nabla^2 \mathbf{E} - \frac{1}{v^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \tag{2.45}$$

$$\nabla^2 \mathbf{B} - \frac{1}{v^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0, \tag{2.46}$$

where the velocity v is given as

$$v = \frac{1}{\sqrt{\varepsilon \mu_0}}. \tag{2.47}$$

Table 2.2: Dielectric Constant and Refractive Index of Selected Materials

Material	Wavelength	ε_r	n
Silicon	1.39 μm	12.2	3.49
Germanium	2.1 μm	16.8	4.10
TiO_2	2.0 μm	5.76	2.4
SiO_2	Visible	2.40	1.55
Window glass	Visible	2.40	1.55
ZnS	Visible	5.43	2.33
CeO_2	Visible	3.81	1.953
CaF_2	Visible	2.06	1.435
MgF_2	Visible	1.91	1.383

Source: American Institute of Physics Handbook, 3rd Ed, McGraw-Hill, New York, 1982.

Comparing with Eq. 2.14, the relation of v with c is

$$\frac{c}{v} = \sqrt{\frac{\varepsilon}{\varepsilon_0}}. \tag{2.48}$$

Defining the relative dielectric constant of the medium as

$$\varepsilon_r \equiv \frac{\varepsilon}{\varepsilon_0}, \tag{2.49}$$

the ratio of the speed of light in a vacuum and the speed of light in the medium, defined as the *refractive index* n, is

$$n \equiv \frac{c}{v} = \sqrt{\varepsilon_r}. \tag{2.50}$$

In general, the relative dielectric constant and the refractive index depend on the frequency or wavelength of the electromagnetic wave. For application in solar energy devices, the most relevant case is solar radiation in the visible or infrared. Table 2.2 shows the relative dielectric constant and refractive index of several materials often used in solar energy devices.

For electromagnetic waves propagating in the z direction with wavevector k and electric field intensity in x, similar to Eqs. 2.24–2.26, the nonzero components are

$$E_x = E_0\, e^{i(kz-\omega t)} \tag{2.51}$$

$$B_y = \frac{k}{\omega} E_0\, e^{i(kz-\omega t)}. \tag{2.52}$$

The wavevector k is given as

$$k = \frac{\omega}{v} = \frac{\omega n}{c}. \tag{2.53}$$

And, according to Eq. 2.50, the electric and magnetic fields are in phase and proportional,

$$B_y = \frac{1}{v}\, E_x = \frac{n}{c}\, E_x. \tag{2.54}$$

2.2.2 Energy Balance and Poynting Vector

Let us study the energy balance in an electromagnetic field by considering a unit volume with relatively uniform fields. If the current density is $\mathbf{J}$ and the electric field intensity is $\mathbf{E}$, the ohmic energy loss per unit time per unit volume is $\mathbf{J} \cdot \mathbf{E}$. Using Eq. 2.44, the expression of energy loss becomes

$$\mathbf{J} \cdot \mathbf{E} = -\frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) + \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}. \tag{2.55}$$

Using the mathematical identity

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\nabla \cdot (\mathbf{E} \times \mathbf{B}) + \mathbf{B} \cdot (\nabla \times \mathbf{E}), \tag{2.56}$$

Eq. 2.55 becomes

$$\mathbf{J}\cdot\mathbf{E} = \nabla\cdot\left(\frac{1}{\mu_0}\,\mathbf{E}\times\mathbf{B}\right) + \frac{1}{\mu_0}\mathbf{B}\cdot(\nabla\times\mathbf{E}) - \varepsilon\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}. \tag{2.57}$$

Using Eq. 2.43, Eq. 2.57 becomes

$$\mathbf{J}\cdot\mathbf{E} = -\nabla\cdot\left(\frac{1}{\mu_0}\,\mathbf{E}\times\mathbf{B}\right) - \frac{\partial}{\partial t}\left(\frac{\varepsilon}{2}E^2 + \frac{1}{2\mu_0}B^2\right). \tag{2.58}$$

The right-hand side of Eq. 2.58 has a straightforward explanation. The energy density of the electromagnetic fields is

$$W = \frac{\varepsilon}{2}E^2 + \frac{1}{2\mu_0}B^2, \tag{2.59}$$

and the power density of the electromagnetic field per unit area is

$$\mathbf{S} = \frac{1}{\mu_0}\,\mathbf{E}\times\mathbf{B}. \tag{2.60}$$

The vector **S** is called the *Poynting vector* after its discoverer.

For an electromagnetic wave, according to Eq. 2.54, $cB_y = nE_x$. The magnitude of the Poynting vector along the direction of propagation is

$$S_z = \frac{n}{\mu_0 c}E_x^2. \tag{2.61}$$

2.2.3 Fresnel Formulas

Consider two media of refractive indices n_1 and n_2 with an interface at $z = 0$, as shown in Fig. 2.3. The incident light is moving in the z direction with wavevector k_{I},

$$k_{\mathrm{I}} = \frac{\omega n_1}{c}. \tag{2.62}$$

The field intensities of the incident light are

$$E_{\mathrm{I}} = I\,e^{i(k_{\mathrm{I}}z-\omega t)}, \tag{2.63}$$

$$B_{\mathrm{I}} = \frac{n_1}{c}I\,e^{i(k_{\mathrm{I}}z-\omega t)}, \tag{2.64}$$

where I is a constant characterizing the intensity of incident light.

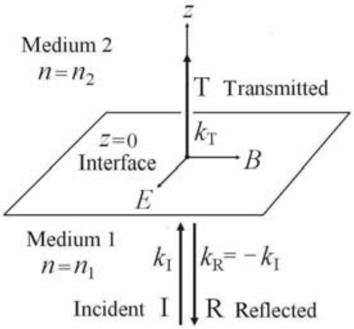
For transmitted light, the wavevector is determined by the refractive index of medium 2,

$$k_{\mathrm{T}} = \frac{\omega n_2}{c}. \tag{2.65}$$

The field intensities of the transmitted light are

$$E_{\mathrm{T}} = T\,e^{i(k_{\mathrm{T}}z-\omega t)}, \tag{2.66}$$

Figure 2.3 Derivation of Fresnel formulas. Two media with indices of refraction n_1 and n_2 share an interface at $z = 0$. The incident light has a wavevector k_I. The wavevector of transmitted light is k_T. The wavevector of reflected light is identical to that of incident light but with opposite sign. By applying Maxwell's equations at the interface, the relations between the three components of light can be derived.



$$B_\mathrm{T} = \frac{n_2}{c}\, T\, e^{i(k_\mathrm{T} z - \omega t)}, \tag{2.67}$$

The constant T characterizing the intensity of the transmitted light is to be determined by the boundary conditions required by Maxwell's equations.

For reflected light, because it is in the same medium as the incident light, the absolute value of the wavevector is identical to that of the incident light. However, the direction of z is reversed. By using the same notation k_I, the field intensities of the reflected light are

$$E_\mathrm{R} = R\, e^{i(-k_\mathrm{I} z - \omega t)}, \tag{2.68}$$

$$B_\mathrm{R} = -\frac{n_1}{c}\, R\, e^{i(-k_\mathrm{I} z - \omega t)}. \tag{2.69}$$

Notice the negative sign of the magnetic field intensity B_R. Again, the constant R characterizes the intensity of the reflected light.

On the interface, $z = 0$, following Eqs. 2.1 and 2.2, both electric field intensity and magnetic field intensity should be continuous. In other words,

$$E_\mathrm{I} + E_\mathrm{R} = E_\mathrm{T}, \tag{2.70}$$

$$B_\mathrm{I} + B_\mathrm{R} = B_\mathrm{T}. \tag{2.71}$$

Using Eqs. 2.63–2.71, we find

$$I - R = T, \tag{2.72}$$

$$n_1(I + R) = n_2\, T. \tag{2.73}$$

The solutions of Eqs.2.72 and 2.73 are

$$R = \frac{n_2 - n_1}{n_2 + n_1}\, I, \tag{2.74}$$

$$T = \frac{2n_1}{n_2 + n_1}\, I. \tag{2.75}$$

Equations 2.74 and 2.75 are the *Fresnel formulas* for the case of normal incidence. Obviously, if $n_1 = n_2$, there is no reflected light, and 100% of incident light is transmitted through the interface.

The power densities of the incident, transmitted, and reflected light can be evaluated using Eqs. 2.74 and 2.75 and the expression of the Poynting vector, Eq. 2.60. For incident light, the magnitude is

$$S_{\mathrm{I}} = \frac{1}{\mu_0} E_{\mathrm{I}} B_{\mathrm{I}} = \frac{n_1}{\mu_0 c} I^2. \tag{2.76}$$

For transmitted light,

$$S_{\mathrm{T}} = \frac{1}{\mu_0} E_{\mathrm{T}} B_{\mathrm{T}} = \frac{n_2}{\mu_0 c} T^2. \tag{2.77}$$

Using Eq. 2.74,

$$S_{\mathrm{T}} = \frac{4n_1 n_2}{(n_1+n_2)^2} \frac{n_1}{\mu_0 c} I^2 = \frac{4n_1 n_2}{(n_1+n_2)^2} S_{\mathrm{I}}. \tag{2.78}$$

A dimensionless coefficient of transmission is defined as

$$\mathcal{T} \equiv \frac{S_{\mathrm{T}}}{S_{\mathrm{I}}} = \frac{4n_1 n_2}{(n_1+n_2)^2}. \tag{2.79}$$

Following Eqs. 2.77 and 2.78, the intensity of reflected light can be determined, and a dimensionless coefficient of reflection is defined as

$$\mathcal{R} \equiv \frac{S_{\mathrm{R}}}{S_{\mathrm{I}}} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2. \tag{2.80}$$

For semiconductors, the reflection loss can be significant. For example, for silicon, $n = 3.49$. The reflection coefficient is

$$\mathcal{R} = \frac{(1-3.49)^2}{(1+3.49)^2} \approx 0.3076. \tag{2.81}$$

More than 30% of light is lost by reflection. To build high-efficiency solar cells, an *antireflection coating* is essential. We will discuss this in Section 9.4.

2.3 Blackbody Radiation

It was known for centuries that a hot body emits radiation. At around 700°C, a body becomes red hot. At even higher temperatures, a body emits much more radiation, and the color changes to orange, yellow, white, and even blue. In the late nineteenth century, in order to understand phenomena related to industry technology such as steel making and incandescent light bulbs, heat radiation became a hot subject for physicists.

Although all hot bodies emit radiation, blackbodies emit the maximum amount of radiation at a given temperature. At equilibrium, radiation emitted must equal radiation absorbed. Therefore, the body that emits the maximum amount also absorbs the maximum amount—which should look black. Practically, a blackbody is constructed by opening a small hole on a large cavity, as shown in Fig. 2.4. Any light ray passing through the hole with area A experiences multiple reflections on the internal surface of the cavity. If the material is not absolutely shiny, after several impingements, the light will eventually be completely absorbed by the cavity. Therefore, the small hole on the large cavity always looks black, which is a good example of a blackbody.

2.3.1 Rayleigh–Jeans Law

The energy density of radiation as a function of its frequency was studied in the late nineteenth century by Lord Rayleigh and then by Sir James Jeans using classical statistical physics. They treated standing electromagnetic waves in a cavity as individual modes, and the modes follow the equal-partition law of Maxwell–Boltzmann statistics.

Consider a closed cubic cavity with reflective inner surfaces of side L. A sinusoidal electromagnetic wave with frequency ν satisfies the following equation:

$$\nabla^2 \mathbf{A} + \frac{4\pi^2\nu^2}{c^2}\mathbf{A} = 0. \tag{2.82}$$

Assuming that the cavity is made of metal. On the walls of the cavity, electrical field intensity vanishes. Therefore, the vector potential vanishes. The general solution of Eq. 2.82 satisfying that condition is

$$\mathbf{A} = \mathbf{A}_0 \sin(k_x x)\, \sin(k_y y)\, \sin(k_z z). \tag{2.83}$$

The wavevectors are defined by

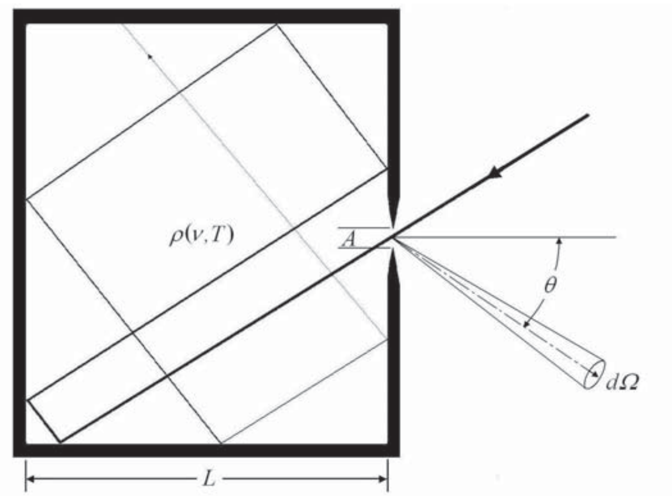


Figure 2.4 Blackbody radiation. A large cavity with a small hole is a good blackbody. The light enters the hole will experience multiple reflections, and all be absorbed and thus looks black. A blackbody emits maximum amount of radiation when heated.

$$k_x = \frac{\pi\, n_x}{L}, \quad k_y = \frac{\pi\, n_y}{L}, \quad k_z = \frac{\pi\, n_z}{L}, \tag{2.84}$$

where n_x, n_y, and n_z are positive integers. By direct substitution one finds that the solution, Eq. 2.83, satisfies differential equation 2.82 and the boundary conditions at the walls. Each set of the integers, n_x, n_y, n_z, represents a pattern of electromagnetic wave in the cavity. Inserting Eq. 2.84 into Eq. 2.82 yields

$$k_x^2 + k_y^2 + k_z^2 = \frac{4\pi^2\nu^2}{c^2}, \tag{2.85}$$

and in terms of the numbers n_x, n_y, and n_z, Eq. 2.85 becomes

$$n_x^2 + n_y^2 + n_z^2 = \frac{4\nu^2 L^2}{c^2}. \tag{2.86}$$

Now, we count the number of standing waves with frequencies ν by considering a sphere of radius $\sqrt{n_x^2 + n_y^2 + n_z^2} = 2\nu L/c$. The number N of modes with positive n_x, n_y, and n_z up to ν is

$$N = \frac{1}{8}\frac{4}{3}\pi \left(\frac{2\nu L}{c}\right)^3 = \frac{4\pi\nu^3\, L^3}{3c^3}, \tag{2.87}$$

For each type of standing wave, there are two polarizations. Therefore, the number of modes of standing electromagnetic waves is

$$N = \frac{8\pi\nu^3\, L^3}{3c^3}, \tag{2.88}$$

where L^3 is the volume, and the *density of states* at frequency ν is

$$\frac{d}{d\nu}\left(\frac{N}{L^3}\right) = \frac{8\pi\nu^2}{c^3}. \tag{2.89}$$

According to Maxwell–Boltzmann statistics, at absolute temperature T, each degree of freedom contributes energy $k_B T$, where k_B is the Boltzmann constant, and the energy density is

$$\rho(\nu, T) = \frac{d}{d\nu}\left(\frac{N}{L^3}\right) k_B T = \frac{8\pi\nu^2}{c^3}\, k_B T. \tag{2.90}$$

Equation 2.90 is the energy density of radiation per unit frequency interval in a cavity of temperature T. It is not directly observable. The directly observable quantity is the spectral radiance $u(\nu, T)$, that is, the energy radiating from a unit area of the hole per unit frequency range. To calculate $u(\nu, T)$ from $\rho(\nu, T)$, first we consider a simplified situation: If the field has a well-defined direction of radiation with velocity c, we have

$$u(\nu, T) = c\,\rho(\nu, T). \tag{2.91}$$

Because the hole is small, the radiation field in a cavity is isotropic. As the radiation only comes through a hole of well-defined direction, $u(\nu, T)$ should be a fraction of $c\rho(\nu, T)$. The value of the fraction can be determined using the following argument. Consider a sphere of radius R. The surface area of the sphere is $4\pi R^2$. If the radiation inside the sphere is allowed to emit over all directions, the area is $4\pi R^2$. If the radiation is allowed to emit in only one direction, the area is a disc with radius R, that is, πR^2. Consequently, the factor is 1/4. Equation 2.91 becomes

$$u(\nu, T) = \frac{1}{4} c\rho(\nu, T). \tag{2.92}$$

Following is a more detailed proof of the factor 1/4. Consider the radiation from a small hole of area A on the cavity; see Fig. 2.4. Because the electromagnetic wave is isotropic and the speed of light is c, the energy radiated through a solid angle $d\Omega$ at an angle θ is

$$\frac{dE}{dt\, d\Omega} = \frac{c}{4\pi} \rho(\nu, T)\, A \cos\theta \tag{2.93}$$

because the area of the hole observed from an angle θ is $A\cos\theta$. Integrating over the hemisphere, the total irradiation per unit area is

$$u(\nu, T) = \frac{c}{4\pi} \int_0^{\pi/2} 2\pi \cos\theta \sin\theta \, d\theta \, \rho(\nu, T) = \frac{c}{4} \rho(\nu, T), \tag{2.94}$$

confirming Eq. 2.92. Using Eq. 2.90, we finally obtain the Rayleigh–Jeans distribution of blackbody radiation,

$$u(\nu, T) = \frac{2\pi\nu^2}{c^2} k_B T. \tag{2.95}$$

The Rayleigh–Jeans distribution fits the low-frequency behavior of the experimental energy density very well. However, as the frequency increases, the spectral irradiance increases, the total irradiation energy is infinite. This contradicts the experimental fact that the total blackbody radiation is finite, and the spectral density has a maximum; see Fig. 2.5.

2.3.2 Planck Formula and Stefan–Boltzmann's Law

In 1900, Max Planck found an empirical formula that fits accurately the experimental data,

$$u(\nu, T) = \frac{2\pi\nu^2}{c^2} \frac{h\nu}{e^{h\nu/k_B T} - 1}. \tag{2.96}$$

The constant h in the formula, Planck's constant, was initially obtained by fitting with experimental blackbody radiation data. Later, Planck found a mathematical explanation of his formula by assuming that the energy of radiation can only take discrete values. Specifically, he assumed that the energy of radiation with frequency ν can only take integer multiples of a basic value $h\nu$, the *energy quantum*,

$$\epsilon = 0,\ \ h\nu,\ \ 2h\nu,\ \ 3h\nu,\ .... \tag{2.97}$$

According to Maxwell–Boltzmann statistics, the probability of finding a state with energy $nh\nu$ is $\exp(-nh\nu/k_BT)$. The average value of energy of a given component of radiation with frequency ν is

$$\bar{\epsilon} = \frac{\sum_{n=0}^{\infty} nh\nu\, e^{-nh\nu/k_BT}}{\sum_{n=0}^{\infty} e^{-nh\nu/k_BT}} = \frac{h\nu}{e^{h\nu/k_BT} - 1}. \tag{2.98}$$

instead of k_BT. By replacing the expression k_BT in Eq. 2.90 with Eq. 2.98, we recovered Eq. 2.96.

Initially, Max Planck believed that the quantization of energy is only a mathematical trick to reconcile his empirically obtained formula with the knowledge of physics known at that time. The profound significance of the concept of quantization of radiation and the meaning of Planck's constant were discovered by Albert Einstein in his interpretation of the photoelectric effect, which is the conceptual foundation of solar cells.

By integrating the spectral radiance over frequency, the total radiation is found to be

$$\begin{aligned} U(T) &= \int_0^{\infty} \frac{2\pi h\nu^3}{c^2} \frac{d\nu}{e^{h\nu/k_BT} - 1} \\ &= \frac{2\pi h}{c^2} \left(\frac{k_BT}{h}\right)^4 \int_0^{\infty} \frac{x^3\, dx}{e^x - 1} \\ &= \frac{2}{15} \frac{\pi^5 k_B^4}{c^2\, h^3}\, T^4. \end{aligned} \tag{2.99}$$

Here a mathematical identity is applied,

$$\int_0^{\infty} \frac{x^3\, dx}{e^x - 1} = \frac{\pi^4}{15}. \tag{2.100}$$

Equation 2.99 is *Stefan–Boltzmann's law*, discovered experimentally before the Planck formula and backed by an argument using thermodynamics. The constant in Eq. 2.99,

$$\sigma \equiv \frac{2}{15} \frac{\pi^5 k_B^4}{c^2 h^3} = \frac{\pi^2 k_B^4}{60\, c^2 \hbar^3} = 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2 \cdot \text{K}^4}, \tag{2.101}$$

is called the *Stefan–Boltzmann's constant*. It can be memorized using the mnemonic: 45678. The total radiance is proportional to the fourth power of absolute temperature, and the coefficient is 5.67 times the inverse eighth power of 10.

For applications in solar cells, the electron volt is the most convenient unit of photon energy; see Fig. 2.5. The Planck formula for blackbody spectral irradiance in terms of photon energy ϵ in units of electron volts is

$$u(\epsilon, T) = \frac{2\pi q^4}{c^2\, h^3}\, \frac{\epsilon^3}{e^{\epsilon/\epsilon_T} - 1} = 1.587 \times 10^8\, \frac{\epsilon^3}{e^{\epsilon/\epsilon_T} - 1}\, \frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{eV}}, \tag{2.102}$$

where $\epsilon_T = k_B T/q$ is the value of $k_B T$ in electron volts. Numerically, it equals $\epsilon_T = T/11,600$. For the Sun, $T_\odot$=5800 K; thus $\epsilon_\odot$ =0.5 eV. At the location of Earth, the radiation is diluted by the distance from the Sun to Earth, the astronomical constant $A_\odot = 1.5 \times 10^{11}$ m. Introducing a geometric factor f representing the solid angle of the Sun with radius $r_\odot = 6.96 \times 10^8$m as observed from Earth

$$f = \left(\frac{r_\odot}{A_\odot}\right)^2 = \frac{\left[6.96 \times 10^8\right]^2}{\left[1.5 \times 10^{11}\right]^2} = 2.15 \times 10^{-5}, \tag{2.103}$$

the spectrum of the AM0 solar radiation (outside the atmosphere at the location of Earth) is

$$u_\oplus(\epsilon, T) = f u_\odot(\epsilon, T) = 3.41 \times 10^3\, \frac{\epsilon^3}{e^{\epsilon/\epsilon_\odot} - 1}\, \frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{eV}}. \tag{2.104}$$

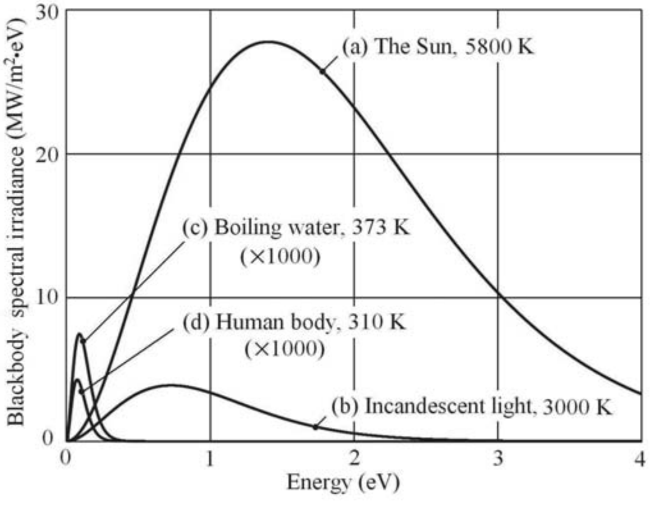


Figure 2.5 Blackbody spectral irradiance. The blackbody spectral irradiance, or the radiation power emitted per square meter per unit energy interval (here in electron volts) at an energy value (also in electron volts) at four different temperatures ise shown. The maximum of solar irradiance is at 1.4 eV, with a value of 27.77 MW/m^2·eV. The temperature of the filament of an incandescent light is about 3000 K. The radiation power density at the filament surface is only about 7% that on the Sun. The spectral irradiance from a blackbody at the boiling points of water and the human body are also shown, in units of kW/m^2·eV.

Table 2.3: Blackbody Radiation at Different Temperatures

Radiator	Temperature (K)	Power (W/m^2)	Peak ϵ (eV)	Peak λ (μm)	Peak u ($W/m^2 \cdot eV$)
The Sun	5800	6.31×10^7	1.410	0.88	2.81×10^7
Light bulb	3000	4.59×10^6	0.728	1.70	3.88×10^6
Boiling water	373	1.10×10^3	0.091	13.6	7.46×10^3
Human body	310	5.24×10^2	0.075	16.5	4.28×10^3

The position of the peak in blackbody spectral irradiance can be of a transcendental equation

$$\frac{d}{dx}\left[3\log x - \log\left(e^x - 1\right)\right] = 0, \tag{2.105}$$

and can be obtained by numerical computation,

$$x = 2.82. \tag{2.106}$$

In other words, the peak of blackbody spectral irradiance is at

$$\epsilon_{\text{MAX}} = 2.82\,\epsilon_T = 2.43 \times 10^{-4}\,T\,(\text{eV}). \tag{2.107}$$

The peak value for the function $x^3/(e^x - 1)$ is 1.42. Therefore, the peak value of the spectral irradiance is

$$u_{\text{MAX}} = 1.42\,\frac{2\pi q k_{\text{B}}^3}{c^2\,h^3}\,T^3 \cong 1.44 \times 10^{-4}\,T^3\;\frac{\text{W}}{\text{m}^2 \cdot \text{eV}}. \tag{2.108}$$

Table 2.3 lists the data for some frequently encountered cases.

2.4 Photoelectric Effect and Concept of Photons

The photoelectric effect was discovered accidentally by Heinrich Hertz in 1887 during experiments to generate electromagnetic waves. Since then, a number of studies have been conducted in an attempt to understand the phenomena. Around 1900, Phillip Lenard did a series of critical studies on the relation of the kinetic energy of ejected electrons with the intensity and wavelength of the impinging light [50]. His results were in direct conflict with the wave theory of light and inspired Albert Einstein to develop his theory of photons.

Figure 2.6 shows schematically the experimental apparatus of Phillip Lenard. The entire setup was enclosed in a vacuum chamber. An electric arc lamp, using carbon rods or zinc rods as the electrodes, generates strong UV light. A quartz window allows such UV light to shine on a target made of different metals. The target and a counter electrode are connected to an adjustable power supply. An ammeter is used to measure the electric current generated by the UV light, the *photocurrent*, especially when the voltages between the two electrodes are very small. By gradually increasing the voltage, which tends to reflect the electrons back to the target, the photocurrent is reduced. The voltage with which the photocurrent becomes zero is recorded as the *stopping voltage*.

The stopping voltage is apparently related to the kinetic energy of the electrons ejected from the target:

$$qV = \frac{1}{2}mv^2. \tag{2.109}$$

Understandably, the photocurrent varies with the intensity of light. By changing the magnitude of the current that drives the arc or the distance from the arc lamp to the target, the photocurrent could change by two orders of magnitude: for example, from 4.1 to 276 pA. An unexpected and dramatic effect Lenard observed was that no matter how strong or how weak the light is, and no matter how large or how small the photocurrent is, the stopping voltage does not change; see Table 2.4. The stopping voltage changes only when the material for the electric arc lamp changes. However, for a given type of arc, the stopping voltage stays unchanged.

The effect Lenard observed has no explanation in the framework of the wave theory of light. According to the wave theory of light, the more intense the light is, the more kinetic energy the electrons acquire.

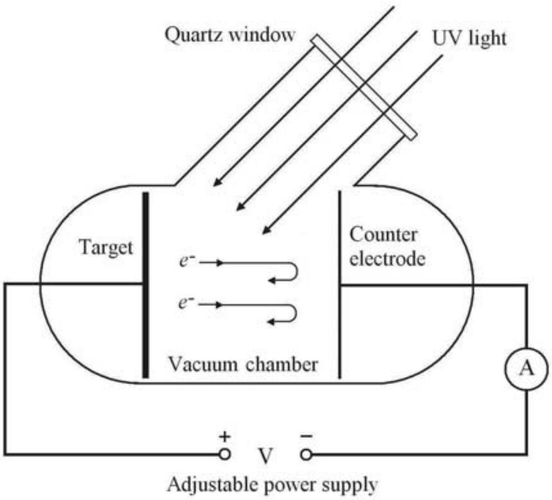


Figure 2.6 Lenard's apparatus for studying photoelectric effect. A quartz window allows the UV light from an electric arc lamp to shine on a target. The voltage between the target and the counter electrode is controlled by an adjustable power supply. An ammeter is used to measure the electric current generated by the UV light, the *photocurrent*. By gradually increasing the voltage (with the polarity as shown), the photocurrent is reduced. The voltage with which the photocurrent becomes zero is recorded as the *stopping voltage* [50].

Table 2.4: Stopping Voltage for Photocurrent

Rod material	Driving current (A)	Distance to target (cm)	Photocurrent (pA)	Stopping voltage (V)
Carbon	28	33.6	276	-1.07
Carbon	20	33.6	174	-1.12
Carbon	28	68	31.7	-1.10
Carbon	8	33.6	4.1	-1.06
Zinc	27	33.6	2180	-0.85
Zinc	27	87.9	319	-0.86

Source: P. Lenard, *Annalen der Physik*, **8,** 167 (1902) [50].

2.4.1 Einstein's Theory of Photons

In 1905, while employed as a patent examiner at the Swiss Patent Office, Albert Einstein wrote five papers, published in *Annalen der Physik*, that initiated the twentieth century revolution in science. For general public, Einstein is mostly known for his theory of relativity. Therefore, when the Swedish Academy announced in 1922 that Einstein had won the Nobel Prize "for services to theoretical physics and especially for the discovery of the law of the photoelectric effect," referring to his paper *On a Heuristic Viewpoint Concerning the Production and Transformation of Light* [27], the public was surprised. In hindsight, the Nobel Committee was correct: His paper on photoelectric effect is considered the boldest, the most revolutionary, and the most original. Although its predictions were fully verified by experiments, for many years, several prominent physicists did not accept Einstein's concept of photons. Here is a quote from Einstein [27]:

> According to the assumption considered here, when a light ray starting from a point is propagated, the energy is not continuously distributed over an ever increasing volume, but it consists of a finite number of energy quanta, localized in space, which move without being divided and which can be absorbed or emitted only as a whole.

According to Einstein, light, when it interacts with matter, appears as a flow of individual and indivisible particles. When a photon interacts with an electron, either it is absorbed or there is no interaction. The energy value of a photon, ϵ, depends on its frequency,

$$\epsilon = h\nu, \tag{2.110}$$

where $h = 6.63 \times 10^{-34}\,\mathrm{J \cdot s}$ is Planck's constant and ν is the frequency of light. For example, for green light, $\lambda = 0.53\,\mu\mathrm{m}$, and the frequency is $5.6 \times 10^{14}\,\mathrm{s}^{-1}$. The energy of the photon is $3.7 \times 10^{-19}\mathrm{J}$, or 2.3 eV.

When a photon interacts with an electron in the metal, it transfers the entire energy to the electron. The electron could escape from the metal by overcoming the *work function* ϕ of the metal, typically a few electron volts. If the energy of the photon is smaller than the work function of the metal, the electron would stay in the metal. If the energy of the photon is greater than the work function of the metal, then the electron can escape from the metal surface *with an excess kinetic energy*,

$$\frac{1}{2}mv^2 = h\nu - \phi. \tag{2.111}$$

The kinetic energy of an escaping electron can be measured by an external voltage, or electric field, to turn it back onto the target. Voltage that just is enough to cancel the kinetic energy is called the *stopping voltage*,

$$q\,V_{\text{stop}} = \frac{1}{2}mv^2 = h\nu - \phi, \tag{2.112}$$

where q is the electron charge, 1.60×10^{-19} C. According to Einstein's quantum theory of light, the stopping voltage is linearly dependent on the frequency of the photon and *independent of the intensity of light*. The slope should be a universal constant, which provides a direct method to determine the value of Planck's constant,

$$\frac{\Delta V_{\text{stop}}}{\Delta \nu} = \frac{h}{q}. \tag{2.113}$$

2.4.2 Millikan's Experimental Verification

Einstein's theory of photons was rejected by a number of prominent physicists for many years, including Max Planck, Niels Bohr, and notably Robert Millikan. Starting in 1905, for 10 years Millikan worked to disprove Einstein's theory. Finally, in 1916, Millikan published a long paper on *Physical Review*, entitled *A Direct Photoelectric Determination of Planck's h* [61]. The conclusion reads as follows:

> 1. Einstein's photoelectric equation has been subject to very searching tests and it appears in every case to predict exactly the observed results.
>
> 2. Planck's h has been photoelectrically determined with a precision of about .5 percent.

In 1923, Millikan received a Nobel Prize "for his work on the elementary charge of electricity and on the photoelectric effect."

An interesting fact in the history of science is that in the same paper Millikan emphatically rejected Einstein's theory of photons. He said that Einstein's photon hypothesis "may well be called reckless first because an electromagnetic disturbance which remains localized in space seems a violation of the very conception of an electromagnetic disturbance, and second because it flies in the face of the thoroughly established facts of interference." Millikan wrote that Einstein's photoelectric equation, although accurately representing the experimental data, "cannot in my judgment be looked upon at

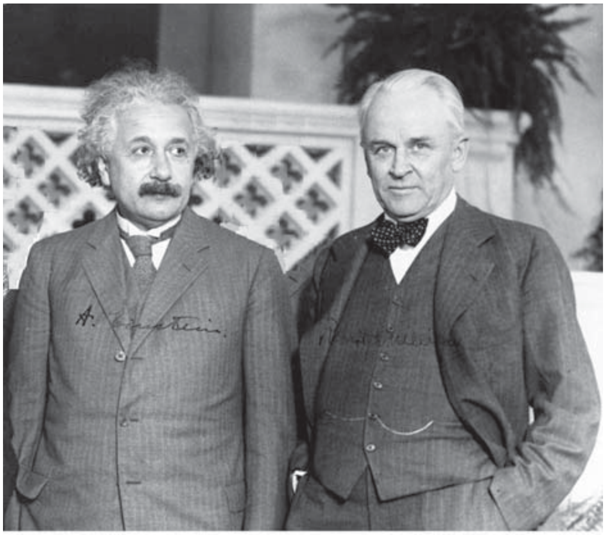


Figure 2.7 Albert Einstein and Robert Millikan. Both Einstein and Millikan won a Nobel Prize for their contributions to the photoelectric effect. Photograph taken in 1930 when Robert Millikan invited Albert Einstein to a conference in California. Original photograph courtesy of Smithsonian Museum, slightly cleaned up by the author.

present as resting upon any sort of a satisfactory theoretical foundation [61].” In 1950, at age 82, in his autobiography [62], Millikan reversed his position and admitted that his experiments

> proved simply and irrefutably, I thought, that the emitted electron that escapes with the energy $h\nu$ gets that energy by the direct transfer of $h\nu$ units of energy from the light to the electron, and hence scarcely permits of any other interpretation than that which Einstein had originally suggested, namely that of the semi-corpuscular or photon theory of light itself.

2.4.3 Wave–Particle Duality

The earlier objections to Einstein’s theory of photons was related to an even more profound problem: the wave–particle duality of all particles. At the beginning of the twentieth century, electrons were described by classical mechanics as being similar to billiard balls. Einstein’s theory seemed to imply that photons are also like billiard balls and that the photoelectric effect is a collision of the billiard balls with electrons. Such a picture is not only hard to conceive but also in direct conflict with the well-established interference phenomenon of light.

The paradox was resolved after Louis de Broglie extended Einstein’s postulate that light can be both wave and particle to *all particles*, including the electron. According to de Broglie, a particle with a momentum **p** should also be a plane wave with a wavevector **k** such that

$$\mathbf{p} = \hbar\,\mathbf{k}, \tag{2.114}$$

where $\hbar = h/2\pi$, Planck's constant divided by 2π, is often called Dirac's constant. According to the theory of de Broglie, a better picture of the photoelectric effect is that light radiation as a plane wave interacts with the electron in the electrode, which is also a plane wave, but the energy transferred to the electron must be quantized to satisfy the Einstein equation

$$\epsilon = h\nu \equiv \hbar\omega, \tag{2.115}$$

where $\omega = 2\pi\nu$ is the circular frequency of the light wave. This is an essential concept regarding the understanding of solar cells and solar photochemistry, which we will discuss in the corresponding chapters.

2.5 Einstein's Derivation of Blackbody Formula

Based on the concept of photons and the interaction of photons with matter, Einstein made a very simple derivation of the blackbody radiation formula. The key of his derivation is the introduction of *stimulated emission*, which gave birth to the laser, an acronym for light amplification by stimulated emission of radiation, and provides a better understanding of the interaction between solar radiation and atomic systems.

Einstein studied a simple two-state atomic system; see Fig. 2.8. The radiation field is represented by an energy density $\rho(\nu)$, where ν is the frequency. The atomic system has two states with an energy difference $h\nu$. According to Maxwell–Boltzmann statistics, the ratio of the populations of the two states is

$$\frac{N_2}{N_1} = e^{-h\nu/k_B T}. \tag{2.116}$$

Einstein assumed three transition coefficients: the absorption coefficient B_{12}, the spontaneous emission coefficient A, and the stimulated emission coefficient B_{21}. The rate equations are

$$\frac{dN_2}{dt} = B_{12}N_1\rho(\nu) - B_{21}N_2\rho(\nu) - AN_2, \tag{2.117}$$

$$\frac{dN_1}{dt} = -B_{12}N_1\rho(\nu) + B_{21}N_2\rho(\nu) + AN_2. \tag{2.118}$$

At equilibrium, both dN_1/dt and dN_2/dt should vanish. Therefore,

$$\frac{N_2}{N_1} = \frac{B_{12}\rho(\nu)}{A + B_{21}\rho(\nu)} = e^{-h\nu/k_B T}. \tag{2.119}$$

The coefficients should not depend on temperature. At high temperature, the power density should be high, and the right-hand side of Eq. 2.119 should approach unity.

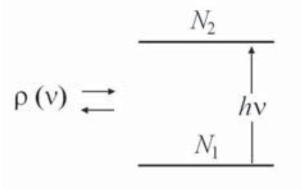


Figure 2.8 Einstein's derivation of blackbody radiation formula. The radiation field $\rho(\nu)$ interacts with a two-level atomic system. Three interaction modes are assumed: absorption, to lift the atomic system from state 1 to state 2; spontaneous emission and stimulated emission, the atomic system decays from state 2 to state 1, giving out energy to the radiation field.

Therefore, one must have

$$B_{12} = B_{21} = B. \tag{2.120}$$

The absorption coefficient B_{12} equals the stimulated emission coefficient B_{21}, which can be represented by a single coefficient B. Under any temperature, the power density distribution of radiation is then

$$\rho(\nu) = \frac{A}{B}\frac{1}{e^{h\nu/k_{\rm B}T} - 1}. \tag{2.121}$$

For radiations of low photon energy, Eq. 2.121 reduces to

$$\rho(\nu) \to \frac{A}{B}\frac{k_{\rm B}T}{h\nu}. \tag{2.122}$$

It should be identical to the Rayleigh–Jeans formula. Comparing with Eq. 2.90, we find the ratio of coefficients A and B,

$$\frac{A}{B} = \frac{8\pi h\nu^3}{c^3}. \tag{2.123}$$

Finally, Planck's formula is recovered,

$$\rho(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/k_{\rm B}T} - 1}. \tag{2.124}$$

Problems

2.1. Show that the capacitance C of a parallel-plate capacitor with vacuum as the dielectric is

$$C = \frac{\varepsilon_0 A}{d}\ [\mathrm{F}], \tag{2.125}$$

where A is the area and d is the distance between the electrodes.

2.2. Show that the capacitance C of a parallel-plate capacitor with a medium of relative dielectric constant ε_r is

$$C = \frac{\varepsilon_0 \varepsilon_r A}{d}\ [\mathrm{F}]. \tag{2.126}$$

Calculate the capacitance of a capacitor with $A = 1\,\mathrm{m}^2$ and $s = 1\,\mathrm{mm}$ for glass and silicon.

2.3. Show that the inductance L of an inductor made of a long solenoid of N loops with cross-sectional area A and length l is

$$L = \frac{\mu_0 N^2 A}{l}\ [\mathrm{H}]. \tag{2.127}$$

2.4. Show that the speed of light v in a medium of relative dielectric constant ε_r is

$$v = \frac{c}{\sqrt{\varepsilon_r}}. \tag{2.128}$$

Calculate the speed of light v in glass and silicon (the relative dielectric constants ε_r for glass and silicon are 2.25 and 11.7, respectively).

2.5. The refractive index of window glass is $n = 1.50$. How much light power is lost when going through a sheet of glass at normal incidence? (*Hint*: there are two glass–air interfaces.)

2.6. The radius of the Sun is $R = 6.96 \times 10^8$ m, and the distance between the Sun and Earth is $D = 1.5 \times 10^{11}$ m. The solar constant is 1366 W/m^2. Estimate the surface temperature of the Sun. (*Hint*: use the Stefan–Boltzmann law.)

2.7. What is the magnitude of the electric field intensity of the sunlight just outside the atmosphere of Earth?

2.8. What is the electric field intensity of the electron in a hydrogen atom at the distance of one Bohr radius from the proton?

2.9. Derive the blackbody radiation spectral density per unit wavelength in unit of micrometers.

2.10. Using the blackbody radiation formula per unit wavelength, derive the Wien displacement law in micrometers.

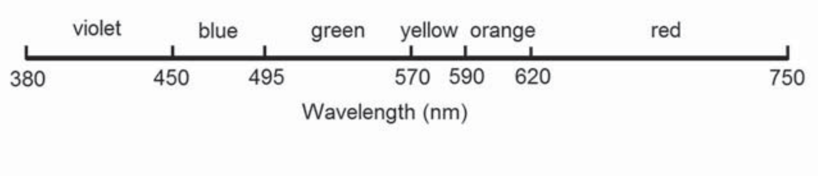


Figure 2.9 Wavelengths of visible lights.

2.11. The wavelengths of visible light with different colors in nanometers are shown in Fig. 2.9. Compute the frequencies and energy values of the photons, in both joules and electron volts.

2.12. What is the solar constant of Venus? Assume that the Sun is a blackbody emitter at 5800 K and the mean Venus-Sun distance is 1.08×10^{11} m.

2.13. To compute the blackbody irradiation for photon energy from ϵ_0 to infinity, an easy-to-use formula can be obtained by introducing $x_0 = \epsilon_0 / k_B T$ and expanding the denominator of Eq. 2.99 into

$$\begin{aligned} U(T, \epsilon_0) &= \frac{2\pi (k_B T)^4}{c^2 h^3} \int_{x_0}^{\infty} \frac{e^{-x} x^3 \, dx}{1 - e^{-x}} \\ &= \frac{2\pi (k_B T)^4}{c^2 h^3} \sum_{n=1}^{\infty} \int_{x_0}^{\infty} e^{-nx} x^3 \, dx. \end{aligned} \tag{2.129}$$

Prove that

$$U(T, \epsilon_0) = \frac{2\pi (k_B T)^4}{c^2 h^3} \sum_{n=1}^{\infty} e^{-n x_0} \left[\frac{x_0^3}{n} + \frac{3x_0^2}{n^2} + \frac{6x_0}{n^3} + \frac{6}{n^4} \right], \tag{2.130}$$

with

$$x_0 = \frac{\epsilon_0}{k_B T}. \tag{2.131}$$

2.14. Assuming that the Sun is a blackbody emitter at 5800 K, what fraction of solar radiation is green (wavelength between 495 and 570 nm)?

2.15. Assuming that the Sun is a blackbody emitter at 5800 K, what fraction of solar radiation has photon energy greater than 1.1 eV?